MATH 135 — QUIZ 8 SOLUTIONS — JAMES HOLLAND
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Question 1. Consider $f(x) = x^3 - 3x^2 - 9x + 7$.

i. What are the critical points of f?
ii. Classify the critical points as relative minimums, maximums, or neither using the second-derivative test.
iii. Classify the critical points as relative minimums, maximums, or neither using the first-derivative test.

Solution .:.

i. $f'(x) = 3x^2 - 6x - 9$. This is always defined. $f'(x) = 0$ iff $x^2 - 2x - 3 = 0$, iff $(x-3)(x+1) = 0$, which happens iff $x = 3$ or $x = -1$. So 3 and -1 are the only critical points.
ii. $f''(x) = 6x - 6$ so that $f''(-1) = -12 < 0$ implies $x = -1$ is a relative maximum, and $f''(3) = 12 > 0$ implies $x = 3$ is a relative minimum.
iii. We need to check whether $f'(x)$ is positive or negative in the three relevant intervals: $(-\infty, -1)$, $(-1, 3)$, and $(3, \infty)$.
   - For $x < -1$, $x - 3$ and $x + 1$ are both negative. Hence $f'(x) = 3(x-3)(x+1)$ is the product of two negative numbers, and is thus positive.
   - For $-1 < x < 3$, $0 < x + 1$ and $x - 3 < 0$. Hence $f'(x) = 3(x-3)(x+1)$ is the product of a positive and a negative number, and is thus negative.
   - For $x > 3$, $x - 3$ and $x + 1$ are both positive, and therefore $f'(x) = 3(x-3)(x+1)$ is positive.

   So by the first-derivative test, $x = -1$ gives a relative maximum and $x = 3$ gives a relative minimum.

Question 2. Consider $f(x) = x\tan(1/x)$.

i. Determine the horizontal asymptotes of f.
ii. Determine the vertical asymptotes of f.

Solution .:.

i. The horizonal asymptotes of f are given by the limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.
   For the first limit, $\lim_{x\to\infty} x\tan(1/x) = \lim_{x\to\infty} \frac{x\sin(1/x)}{\cos(1/x)}$. As x goes to ∞, $t = 1/x$ goes to 0. So this limit is the same as
   $$\lim_{x\to\infty} \frac{\sin(1/x)}{\frac{1}{x}\cos(1/x)} = \lim_{t\to 0} \frac{\sin(t)}{t} \cdot \cos(t) = 1 \cdot 1 = 1.$$
   For the second limit, the same reasoning applies since $\lim_{x\to-\infty} 1/x = 0$ so that $\lim_{x\to-\infty} x\tan(1/x) = 1$. Thus $y = 1$ is the only horizontal asymptote of f.
ii. $f(x)$ is undefined iff $x\sin(1/x)/\cos(1/x)$ is undefined, which happens iff $\cos(1/x) = 0$. This requires $1/x = \pi n$ for some integer n, meaning $x = \frac{1}{\pi n}$ for some integer n. These are the vertical asymptotes since $\lim_{x\to 1/(\pi n)} x\sin(1/x) = \frac{1}{\pi n} \cdot \pm 1 \neq 0$.